

An S -type eigenvalue localization set for tensors

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Abstract

An S -type eigenvalue localization set for a tensor is given by breaking $N = \{1, 2, \dots, n\}$ into disjoint subsets S and its complement. It is shown that the new set is tighter than those provided by L. Qi (Journal of Symbolic Computation 40 (2005) 1302-1324) and Li et al. (Numer. Linear Algebra Appl. 21 (2014) 39-50). As applications of the results, a checkable sufficient condition for the positive definiteness of tensors and a checkable sufficient condition of the positive semi-definiteness of tensors are given.

Keywords: Tensor eigenvalue, Localization set, Positive definite, Positive semi-definite

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1. Introduction

Eigenvalue problems of tensors have become an important topic of study in numerical multilinear algebra, and they have a wide range of practical applications; see [2, 3, 4, 5, 7, 8, 9, 10, 11, 17, 18, 19, 21, 22, 24, 25]. Here we call $\mathcal{A} = (a_{i_1 \dots i_m})$ a complex (real) tensor of order m dimension n , denoted by $\mathcal{A} \in C^{[m,n]} (R^{[m,n]})$, if

$$a_{i_1 \dots i_m} \in C (R),$$

where $i_j = 1, \dots, n$ for $j = 1, \dots, m$. Moreover, if there are a complex number λ and a nonzero complex vector $x = (x_1, x_2, \dots, x_n)^T$ such that

$$\mathcal{A}x^{m-1} = \lambda x^{[m-1]},$$

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then λ is called an eigenvalue of \mathcal{A} and x an eigenvector of \mathcal{A} associated with λ [16, 20], where $\mathcal{A}x^{m-1}$ is an n dimension vector whose i th component is

$$(\mathcal{A}x^{m-1})_i = \sum_{i_2, \dots, i_m \in N} a_{ii_2 \dots i_m} x_{i_2} \cdots x_{i_m} \quad (N = \{1, 2, \dots, n\})$$

and

$$x^{[m-1]} = (x_1^{m-1}, x_2^{m-1}, \dots, x_n^{m-1})^T.$$

If x and λ are all real, then λ is called an H-eigenvalue of \mathcal{A} and x an H-eigenvector of \mathcal{A} associated with λ [20, 21, 27].

One of many practical applications of eigenvalues of tensors is that one can identify the positive (semi-)definiteness for an even-order real symmetric tensor by using the smallest H-eigenvalue of a tensor, consequently, can identify the positive (semi-)definiteness of the multivariate homogeneous polynomial determined by this tensor, for details, see [11, 20].

Because it is not easy to compute the smallest H-eigenvalue of tensors when the order and dimension are large, ones always try to give a set including all eigenvalues in the complex [20, 13, 14, 15]. In particular, if this set for an even-order real symmetric tensor is in the right-half complex plane, then we can conclude that the smallest H-eigenvalue is positive, consequently, the corresponding tensor is positive definite.

In [20], Qi gave an eigenvalue localization set for real symmetric tensors, which is a generalization of the well-known Geršgorin's eigenvalue localization set of matrices [6, 23]. This result can be easily generalized to general tensors [13, 26].

Theorem 1. [13, 20, 26] Let $\mathcal{A} = (a_{i_1 \dots i_m}) \in C^{[m, n]}$. Then

$$\sigma(\mathcal{A}) \subseteq \Gamma(\mathcal{A}) := \bigcup_{i \in N} \Gamma_i(\mathcal{A}),$$

where $\sigma(\mathcal{A})$ is the set of all the eigenvalues of \mathcal{A} ,

$$\Gamma_i(\mathcal{A}) = \{z \in \mathbb{C} : |z - a_{i \dots i}| \leq r_i(\mathcal{A})\}, \quad r_i(\mathcal{A}) = \sum_{\substack{i_2, \dots, i_m \in N, \\ \delta_{ii_2 \dots i_m} = 0}} |a_{ii_2 \dots i_m}|$$

and

$$\delta_{i_1 \dots i_m} = \begin{cases} 1, & \text{if } i_1 = \dots = i_m, \\ 0, & \text{otherwise.} \end{cases}$$

Although it is easy to get $\Gamma(\mathcal{A})$ in the complex by computing n sets $\Gamma_i(\mathcal{A})$, $\Gamma(\mathcal{A})$ doesn't always capture all eigenvalues of \mathcal{A} very precisely. To obtain tighter sets than $\Gamma(\mathcal{A})$, Li et al. [13] extended the Brauer's eigenvalue localization set of matrices [1, 23] and gave the following Brauer-type eigenvalue localization set for tensors.

Theorem 2. [13] *Let $\mathcal{A} = (a_{i_1 \dots i_m}) \in C^{[m, n]}$, $n \geq 2$. Then*

$$\sigma(\mathcal{A}) \subseteq \mathcal{K}(\mathcal{A}) := \bigcup_{\substack{i, j \in N, \\ j \neq i}} \mathcal{K}_{i, j}(\mathcal{A}),$$

where

$$\mathcal{K}_{i, j}(\mathcal{A}) = \{z \in \mathbb{C} : (|z - a_{i \dots i}| - r_i^j(\mathcal{A})) |z - a_{j \dots j}| \leq |a_{ij \dots j}| r_j(\mathcal{A})\}$$

and

$$r_i^j(\mathcal{A}) = \sum_{\substack{\delta_{i, i_2, \dots, i_m} = 0, \\ \delta_{j, i_2, \dots, i_m} = 0}} |a_{ii_2 \dots i_m}| = r_i(\mathcal{A}) - |a_{ij \dots j}|.$$

Furthermore, $\mathcal{K}(\mathcal{A}) \subseteq \Gamma(\mathcal{A})$.

As Theorem 2 shows, we need compute $n(n-1)$ sets $\mathcal{K}_{i, j}(\mathcal{A})$ to give the set $\mathcal{K}(\mathcal{A})$, however $\mathcal{K}(\mathcal{A})$ captures all eigenvalues of \mathcal{A} more precisely than $\Gamma(\mathcal{A})$. To reduce computations, Li et al. give an S -type eigenvalue localization set by breaking N into disjoint subsets S and \bar{S} , where \bar{S} is the complement of S in N .

Theorem 3. [13, Theorem 2.2] *Let $\mathcal{A} = (a_{i_1 \dots i_m}) \in C^{[m, n]}$, $n \geq 2$, and S be a nonempty proper subset of N . Then*

$$\sigma(\mathcal{A}) \subseteq \mathcal{K}^S(\mathcal{A}) := \left(\bigcup_{\substack{i \in S, \\ j \neq \bar{S}}} \mathcal{K}_{i, j}(\mathcal{A}) \right) \cup \left(\bigcup_{\substack{i \in \bar{S}, \\ j \neq S}} \mathcal{K}_{i, j}(\mathcal{A}) \right).$$

The set $\mathcal{K}^S(\mathcal{A})$ in Theorem 3 consists of $2|S|(n - |\bar{S}|)$ sets $\mathcal{K}_{i, j}(\mathcal{A})$, where $|S|$ is the cardinality of S . It is obvious that $2|S|(n - |\bar{S}|) \leq n(n-1)$, and then

$$\mathcal{K}^S(\mathcal{A}) \subseteq \mathcal{K}(\mathcal{A}) \subseteq \Gamma(\mathcal{A}), \quad (1)$$

for details, see [13]. In this paper, by the technique in [13] we give a new eigenvalue localization set involved with a proper subset S of N , and prove that the new set is tighter than $\Gamma(\mathcal{A})$, $\mathcal{K}(\mathcal{A})$ and $\mathcal{K}^S(\mathcal{A})$. As an application, we give some checkable sufficient conditions for the positive (semi-)definiteness of tensors.

2. A new S -type eigenvalue localization set

we begin with some notation. Given an nonempty proper subset S of N , we denote

$$\Delta^N := \{(i_2, i_3, \dots, i_m) : \text{each } i_j \in N \text{ for } j = 2, \dots, m\},$$

$$\Delta^S := \{(i_2, i_3, \dots, i_m) : \text{each } i_j \in S \text{ for } j = 2, \dots, m\},$$

and then

$$\overline{\Delta^S} = \Delta^N \setminus \Delta^S.$$

This implies that for a tensor $\mathcal{A} = (a_{i_1 \dots i_m}) \in C^{[m, n]}$, we have that for $i \in S$,

$$r_i(\mathcal{A}) = r_i^{\Delta^S}(\mathcal{A}) + r_i^{\overline{\Delta^S}}(\mathcal{A}), \quad r_i^j(\mathcal{A}) = r_i^{\Delta^S}(\mathcal{A}) + r_i^{\overline{\Delta^S}}(\mathcal{A}) - |a_{ij \dots j}|,$$

where

$$r_i^{\Delta^S}(\mathcal{A}) = \sum_{\substack{(i_2, \dots, i_m) \in \Delta^S, \\ \delta_{ii_2 \dots i_m} = 0}} |a_{ii_2 \dots i_m}|, \quad r_i^{\overline{\Delta^S}}(\mathcal{A}) = \sum_{(i_2, \dots, i_m) \in \overline{\Delta^S}} |a_{ii_2 \dots i_m}|$$

Theorem 4. *Let $\mathcal{A} = (a_{i_1 \dots i_m}) \in C^{[m, n]}$, $n \geq 2$, and S be a nonempty proper subset of N . Then*

$$\sigma(\mathcal{A}) \subseteq \Omega^S(\mathcal{A}) := \left(\bigcup_{\substack{i \in S, \\ j \in \bar{S}}} \Omega_{i,j}^S(\mathcal{A}) \right) \cup \left(\bigcup_{\substack{i \in \bar{S}, \\ j \in S}} \Omega_{i,j}^{\bar{S}}(\mathcal{A}) \right),$$

where

$$\Omega_{i,j}^S(\mathcal{A}) := \left\{ z \in \mathbb{C} : (|\lambda - a_{i \dots i}|) \left(|\lambda - a_{j \dots j}| - r_j^{\overline{\Delta^S}}(\mathcal{A}) \right) \leq r_i(\mathcal{A}) r_j^{\Delta^S}(\mathcal{A}) \right\}$$

and

$$\Omega_{i,j}^{\bar{S}}(\mathcal{A}) := \left\{ z \in \mathbb{C} : (|\lambda - a_{i \dots i}|) \left(|\lambda - a_{j \dots j}| - r_j^{\overline{\Delta^S}}(\mathcal{A}) \right) \leq r_i(\mathcal{A}) r_j^{\Delta^S}(\mathcal{A}) \right\}.$$

Proof. For any $\lambda \in \sigma(\mathcal{A})$, let $x = (x_1, x_2, \dots, x_n)^T \in \mathbb{C}^n \setminus \{0\}$ be an associated eigenvector, i.e.,

$$\mathcal{A}x^{m-1} = \lambda x^{[m-1]}. \quad (2)$$

Let $|x_p| = \max_{i \in S} |x_i|$ and $|x_q| = \max_{i \in \bar{S}} |x_i|$. Then, at least one of x_p and x_q is nonzero. We next divide into three cases to prove.

Case I: $x_p x_q \neq 0$ and $|x_q| \geq |x_p|$, that is, $|x_q| = \max_{i \in N} |x_i|$. By (2), we have

$$(\lambda - a_{q \dots q})x_q^{m-1} = \sum_{(i_2, \dots, i_m) \in \Delta^S} a_{qi_2 \dots i_m} x_{i_2} \cdots x_{i_m} + \sum_{\substack{(i_2, \dots, i_m) \in \overline{\Delta^S}, \\ \delta_{qi_2 \dots i_m} = 0}} a_{qi_2 \dots i_m} x_{i_2} \cdots x_{i_m}.$$

Taking modulus in the above equation and using the triangle inequality gives

$$\begin{aligned} |\lambda - a_{q \dots q}| |x_q|^{m-1} &\leq \sum_{(i_2, \dots, i_m) \in \Delta^S} |a_{qi_2 \dots i_m}| |x_{i_2}| \cdots |x_{i_m}| + \sum_{\substack{(i_2, \dots, i_m) \in \overline{\Delta^S}, \\ \delta_{qi_2 \dots i_m} = 0}} |a_{qi_2 \dots i_m}| |x_{i_2}| \cdots |x_{i_m}| \\ &\leq \sum_{(i_2, \dots, i_m) \in \Delta^S} |a_{qi_2 \dots i_m}| |x_p|^{m-1} + \sum_{\substack{(i_2, \dots, i_m) \in \overline{\Delta^S}, \\ \delta_{qi_2 \dots i_m} = 0}} |a_{qi_2 \dots i_m}| |x_q|^{m-1} \\ &= r_q^{\Delta^S}(\mathcal{A}) |x_p|^{m-1} + r_q^{\overline{\Delta^S}}(\mathcal{A}) |x_q|^{m-1}. \end{aligned}$$

Hence,

$$\left(|\lambda - a_{q \dots q}| - r_q^{\overline{\Delta^S}}(\mathcal{A}) \right) |x_q|^{m-1} \leq r_q^{\Delta^S}(\mathcal{A}) |x_p|^{m-1}. \quad (3)$$

On the other hand, by (2), we also get that

$$(\lambda - a_{p \dots p})x_p^{m-1} = \sum_{\substack{i_2, \dots, i_m \in N, \\ \delta_{pi_2 \dots i_m} = 0}} a_{pi_2 \dots i_m} x_{i_2} \cdots x_{i_m}$$

and

$$|\lambda - a_{p \dots p}| |x_p|^{m-1} \leq r_p(\mathcal{A}) |x_q|^{m-1}. \quad (4)$$

Multiplying (3) with (4) gives

$$(|\lambda - a_{p \dots p}|) \left(|\lambda - a_{q \dots q}| - r_q^{\overline{\Delta^S}}(\mathcal{A}) \right) \leq r_p(\mathcal{A}) r_q^{\Delta^S}(\mathcal{A}),$$

which leads to $\lambda \in \Omega_{p,q}^S(\mathcal{A}) \subseteq \Omega^S(\mathcal{A})$.

Case II: $x_p x_q \neq 0$ and $|x_p| \geq |x_q|$, that is, $|x_p| = \max_{i \in N} |x_i|$. Similar to the proof of Case I, we can obtain that

$$\left(|\lambda - a_{p \dots p}| - r_p^{\overline{\Delta^S}}(\mathcal{A}) \right) |x_p|^{m-1} \leq r_p^{\Delta^S}(\mathcal{A}) |x_q|^{m-1},$$

and

$$|\lambda - a_{q \dots q}| |x_q|^{m-1} \leq r_q(\mathcal{A}) |x_p|^{m-1}.$$

This gives

$$(|\lambda - a_{q\dots q}|) \left(|\lambda - a_{p\dots p}| - r_p^{\overline{\Delta S}}(\mathcal{A}) \right) \leq r_q(\mathcal{A}) r_p^{\Delta \bar{S}}(\mathcal{A}).$$

Hence, $\lambda \in \Omega_{q,p}^{\bar{S}}(\mathcal{A}) \subseteq \Omega^S(\mathcal{A})$.

Case III: $|x_p||x_q| = 0$, without loss of generality, let $|x_p| = 0$ and $|x_q| \neq 0$. Then by (3),

$$|\lambda - a_{q\dots q}| - r_q^{\overline{\Delta S}}(\mathcal{A}) \leq 0.$$

hence for any $i \in S$,

$$(|\lambda - a_{i\dots i}|) \left(|\lambda - a_{q\dots q}| - r_q^{\overline{\Delta S}}(\mathcal{A}) \right) \leq r_i(\mathcal{A}) r_q^{\Delta S}(\mathcal{A}),$$

which leads to $\lambda \in \Omega_{i,q}^S(\mathcal{A}) \subseteq \Omega^S(\mathcal{A})$. The conclusion follows from Cases I, II and III. \square

To compare the sets $\Gamma(\mathcal{A})$ in Theorem 1, $\mathcal{K}(\mathcal{A})$ in Theorem 2, $\mathcal{K}^S(\mathcal{A})$ in Theorem 3 with $\Omega^S(\mathcal{A})$ in Theorem 4, two lemmas in [14] are listed as follows.

Lemma 5. [14, Lemmas 2.2 and 2.3] (I) Let $a, b, c \geq 0$ and $d > 0$. If $\frac{a}{b+c+d} \leq 1$, then

$$\frac{a - (b + c)}{d} \leq \frac{a - b}{c + d} \leq \frac{a}{b + c + d}.$$

(II) Let $a, b, c \geq 0$ and $d > 0$. If $\frac{a}{b+c+d} \geq 1$, then

$$\frac{a - (b + c)}{d} \geq \frac{a - b}{c + d} \geq \frac{a}{b + c + d}.$$

Theorem 6. Let $\mathcal{A} = (a_{i_1\dots i_m}) \in C^{[m,n]}$, $n \geq 2$. And let S be a nonempty proper subset of N . Then

$$\Omega^S(\mathcal{A}) \subseteq \mathcal{K}^S(\mathcal{A}) \subseteq \mathcal{K}(\mathcal{A}) \subseteq \Gamma(\mathcal{A}).$$

Proof. By (1), $\mathcal{K}^S(\mathcal{A}) \subseteq \mathcal{K}(\mathcal{A}) \subseteq \Gamma(\mathcal{A})$ holds. We only prove $\Omega^S(\mathcal{A}) \subseteq \mathcal{K}^S(\mathcal{A})$. Let $z \in \Omega^S(\mathcal{A})$. Then

$$z \in \bigcup_{\substack{i \in S, \\ j \in \bar{S}}} \Omega_{i,j}^S(\mathcal{A}) \text{ or } z \in \bigcup_{\substack{i \in \bar{S}, \\ j \in S}} \Omega_{i,j}^{\bar{S}}(\mathcal{A}).$$

Without loss of generality, suppose that $z \in \bigcup_{\substack{i \in S, \\ j \in \bar{S}}} \Omega_{i,j}^S(\mathcal{A})$ (we can prove it similarly if $z \in \bigcup_{\substack{i \in \bar{S}, \\ j \in S}} \Omega_{i,j}^{\bar{S}}(\mathcal{A})$). Then there are $p \in S$ and $q \in \bar{S}$ such that $z \in \Omega_{p,q}^S(\mathcal{A})$, i.e.,

$$(|z - a_{p\dots p}|) \left(|z - a_{q\dots q}| - r_q^{\bar{\Delta}^S}(\mathcal{A}) \right) \leq r_p(\mathcal{A}) r_q^{\Delta^S}(\mathcal{A}). \quad (5)$$

If $r_p(\mathcal{A}) r_q^{\Delta^S}(\mathcal{A}) = 0$, then $r_p(\mathcal{A}) = 0$, or $r_q^{\Delta^S}(\mathcal{A}) = 0$. When $r_q^{\Delta^S}(\mathcal{A}) = 0$, we have $|a_{qp\dots p}| = 0$, $r_q^{\Delta^S}(\mathcal{A}) = r_q^p(\mathcal{A})$ and

$$\begin{aligned} |z - a_{p\dots p}| \left(|z - a_{q\dots q}| - r_q^p(\mathcal{A}) \right) &= |z - a_{p\dots p}| \left(|z - a_{q\dots q}| - r_q^{\bar{\Delta}^S}(\mathcal{A}) \right) \\ &\leq r_p(\mathcal{A}) r_q^{\Delta^S}(\mathcal{A}) \\ &= r_p(\mathcal{A}) |a_{qp\dots p}| \\ &= 0, \end{aligned}$$

which implies that $z \in \mathcal{K}_{q,p}(\mathcal{A}) \subseteq \bigcup_{\substack{i \in \bar{S}, \\ j \in S}} \mathcal{K}_{i,j}(\mathcal{A}) \subseteq \mathcal{K}^S(\mathcal{A})$, consequently, $\Omega^S(\mathcal{A}) \subseteq \mathcal{K}^S(\mathcal{A})$. When $r_p(\mathcal{A}) = 0$, we have

$$\begin{aligned} |z - a_{p\dots p}| \left(|z - a_{q\dots q}| - r_q^p(\mathcal{A}) \right) &\leq |z - a_{p\dots p}| \left(|z - a_{q\dots q}| - r_q^{\bar{\Delta}^S}(\mathcal{A}) \right) \\ &\leq r_p(\mathcal{A}) r_q^{\Delta^S}(\mathcal{A}) \\ &= 0 \\ &= r_p(\mathcal{A}) |a_{qp\dots p}|. \end{aligned}$$

This also leads to $z \in \bigcup_{\substack{i \in \bar{S}, \\ j \in S}} \mathcal{K}_{i,j}(\mathcal{A}) \subseteq \mathcal{K}^S(\mathcal{A})$, and $\Omega^S(\mathcal{A}) \subseteq \mathcal{K}^S(\mathcal{A})$.

If $r_p(\mathcal{A}) r_q^{\Delta^S}(\mathcal{A}) > 0$, then we can equivalently express Inequality (5) as

$$\frac{|z - a_{q\dots q}| - r_q^{\bar{\Delta}^S}(\mathcal{A})}{r_q^{\Delta^S}(\mathcal{A})} \frac{|z - a_{p\dots p}|}{r_p(\mathcal{A})} \leq 1, \quad (6)$$

which implies

$$\frac{|z - a_{q\dots q}| - r_q^{\bar{\Delta}^S}(\mathcal{A})}{r_q^{\Delta^S}(\mathcal{A})} \leq 1, \quad (7)$$

or

$$\frac{|z - a_{p\dots p}|}{r_p(\mathcal{A})} \leq 1. \quad (8)$$

Let $a = |z - a_{t\dots t}|$, $b = r_q^{\overline{\Delta^S}}(\mathcal{A})$, $c = r_q^{\Delta^S}(\mathcal{A}) - |a_{qp\dots p}|$ and $d = |a_{qp\dots p}|$. When Inequality (7) holds and $d = |a_{qp\dots p}| > 0$, from the part (I) in Lemma 5 and Inequality (6) we have

$$\frac{|z - a_{q\dots q}| - r_q^p(\mathcal{A})}{|a_{qp\dots p}|} \frac{|z - a_{p\dots p}|}{r_p(\mathcal{A})} \leq \frac{|z - a_{q\dots q}| - r_q^{\overline{\Delta^S}}(\mathcal{A})}{r_q^{\Delta^S}(\mathcal{A})} \frac{|z - a_{p\dots p}|}{r_p(\mathcal{A})} \leq 1$$

equivalently,

$$|z - a_{p\dots p}| (|z - a_{q\dots q}| - r_q^p(\mathcal{A})) \leq r_p(\mathcal{A}) |a_{qp\dots p}|.$$

This implies $\Omega^S(\mathcal{A}) \subseteq \mathcal{K}^S(\mathcal{A})$. When Inequality (7) holds and $d = |a_{qp\dots p}| = 0$, we easily get

$$|z - a_{q\dots q}| - r_q^p(\mathcal{A}) \leq 0 = |a_{qp\dots p}|.$$

Hence,

$$|z - a_{p\dots p}| (|z - a_{q\dots q}| - r_q^p(\mathcal{A})) \leq 0 = r_p(\mathcal{A}) |a_{qp\dots p}|.$$

This also implies $\Omega^S(\mathcal{A}) \subseteq \mathcal{K}^S(\mathcal{A})$. On the other hand, when Inequality (8) holds, we only need to prove $\Omega^S(\mathcal{A}) \subseteq \mathcal{K}^S(\mathcal{A})$ under the case that

$$\frac{|z - a_{q\dots q}| - r_q^{\overline{\Delta^S}}(\mathcal{A})}{r_q^{\Delta^S}(\mathcal{A})} > 1. \quad (9)$$

Note that Inequality (9) yields

$$\frac{|z - a_{q\dots q}|}{r_q(\mathcal{A})} > 1.$$

Hence, when Inequality (8) holds and $|a_{pq\dots q}| > 0$, we have from Lemma 5 and Inequality (6) that

$$\frac{|z - a_{q\dots q}|}{r_q(\mathcal{A})} \frac{|z - a_{p\dots p}| - r_p^q(\mathcal{A})}{|a_{pq\dots q}|} \leq \frac{|z - a_{q\dots q}| - r_q^{\overline{\Delta^S}}(\mathcal{A})}{r_q^{\Delta^S}(\mathcal{A})} \frac{|z - a_{p\dots p}|}{r_p(\mathcal{A})} \leq 1$$

equivalently,

$$|z - a_{q\dots q}| (|z - a_{p\dots p}| - r_p^q(\mathcal{A})) \leq r_q(\mathcal{A}) |a_{pq\dots q}|.$$

This implies $z \in \mathcal{K}_{p,q}(\mathcal{A}) \subseteq \bigcup_{\substack{i \in S, \\ j \in \bar{S}}} \mathcal{K}_{i,j}(\mathcal{A}) \subseteq \mathcal{K}^S(\mathcal{A})$ and $\Omega^S(\mathcal{A}) \subseteq \mathcal{K}^S(\mathcal{A})$. And when Inequality (8) holds and $|a_{pq\dots q}| = 0$, we easily get

$$|z - a_{p\dots p}| - r_p^q(\mathcal{A}) \leq 0 = |a_{pq\dots q}|.$$

Hence,

$$|z - a_{q\dots q}| (|z - a_{p\dots p}| - r_p^q(\mathcal{A})) \leq 0 = r_q(\mathcal{A}) |a_{pq\dots q}|.$$

This also implies $\Omega^S(\mathcal{A}) \subseteq \mathcal{K}^S(\mathcal{A})$. \square

Remark 1. For a complex tensor $\mathcal{A} \in C^{[m,n]}$, $n \geq 2$, the set $\mathcal{K}^S(\mathcal{A})$ consists of $2|S|(n - |S|)$ sets $\mathcal{K}_{i,j}(\mathcal{A})$, and the set $\Omega^S(\mathcal{A})$ consists of $|S|(n - |S|)$ sets $\Omega_{i,j}^S(\mathcal{A})$ and $|S|(n - |S|)$ sets $\Omega_{i,j}^{\bar{S}}(\mathcal{A})$, where S is a nonempty proper subset of N . Hence, under the same computations, $\Omega^S(\mathcal{A})$ captures all eigenvalues of \mathcal{A} more precisely than $\mathcal{K}^S(\mathcal{A})$.

3. Sufficient conditions for positive (semi-)definiteness of tensors

As applications of the results in Sections 2, we in this section provide some checkable sufficient conditions for the positive definiteness and positive semi-definiteness of tensors, respectively. Before that, we give some definitions in [5, 12, 28].

Definition 1. [5, 28] A tensor $\mathcal{A} = (a_{i_1\dots i_m}) \in C^{[m,n]}$ is called a (strictly) diagonally dominant tensor if for $i \in N$,

$$|a_{i\dots i}| \geq (>) r_i(\mathcal{A}). \quad (10)$$

Definition 2. [12] A tensor $\mathcal{A} = (a_{i_1\dots i_m}) \in C^{[m,n]}$ with $n \geq 2$ is called a quasi-doubly (strictly) diagonally dominant tensor if for $i, j \in N$, $j \neq i$,

$$(|a_{i\dots i}| - r_i^j(\mathcal{A})) |a_{j\dots j}| \geq (>) r_j(\mathcal{A}) |a_{ij\dots j}|. \quad (11)$$

Definition 3. Let $\mathcal{A} = (a_{i_1\dots i_m}) \in C^{[m,n]}$ with $n \geq 2$ and S be a nonempty proper subset of N . \mathcal{A} is called an S -QDSDD₀ (S -QDSDD) tensor if for each $i \in S$ and each $j \in \bar{S}$, Inequality (11) holds and

$$(|a_{j\dots j}| - r_j^i(\mathcal{A})) |a_{i\dots i}| \geq (>) r_i(\mathcal{A}) |a_{ji\dots j}|. \quad (12)$$

Definition 4. Let $\mathcal{A} = (a_{i_1 \dots i_m}) \in C^{[m,n]}$ with $n \geq 2$ and S be a nonempty proper subset of N . \mathcal{A} is called an S - SDD_0 (S - SDD) tensor if for each $i \in S$ and each $j \in \bar{S}$,

$$|a_{i \dots i}| \left(|a_{j \dots j}| - r_j^{\bar{\Delta}^S}(\mathcal{A}) \right) \geq (>) r_i(\mathcal{A}) r_j^{\Delta^S}(\mathcal{A}), \quad (13)$$

and

$$|a_{j \dots j}| \left(|a_{i \dots i}| - r_i^{\bar{\Delta}^S}(\mathcal{A}) \right) \geq (>) r_j(\mathcal{A}) r_i^{\Delta^S}(\mathcal{A}). \quad (14)$$

Next, we give the relationships between (strictly) diagonally dominant tensors, quasi-doubly (strictly) diagonally dominant tensors, S - $QDSDD_0$ (S - $QDSDD$) tensors and S - SDD_0 (S - SDD) tensors.

Proposition 1. If $\mathcal{A} = (a_{i_1 \dots i_m}) \in C^{[m,n]}$ is a strictly diagonally dominant tensor, then \mathcal{A} is a quasi-doubly strictly diagonally dominant tensor. If \mathcal{A} is a diagonally dominant tensor, then \mathcal{A} is a quasi-doubly diagonally dominant tensor.

Proof. If \mathcal{A} is a strictly diagonally dominant tensor, then for any $i \in N$,

$$|a_{i \dots i}| > r_i(\mathcal{A}),$$

equivalently,

$$|a_{i \dots i}| - r_i^j(\mathcal{A}) > |a_{ij \dots j}|.$$

Hence, for $i, j \in N$, $j \neq i$,

$$|a_{i \dots i}| > r_i(\mathcal{A}),$$

and

$$|a_{j \dots j}| - r_j^i(\mathcal{A}) > |a_{ji \dots i}|,$$

which implies that the strict inequality (11) holds, i.e., \mathcal{A} is a quasi-doubly strictly diagonally dominant tensor by Definition 2. Similarly, we can prove that if \mathcal{A} is a diagonally dominant tensor, then \mathcal{A} is a quasi-doubly diagonally dominant tensor. \square

Proposition 2. If $\mathcal{A} = (a_{i_1 \dots i_m}) \in C^{[m,n]}$ is a quasi-doubly strictly diagonally dominant tensor, then \mathcal{A} is an S - $QDSDD$ tensor. If \mathcal{A} is a quasi-doubly diagonally dominant tensor, then \mathcal{A} is an S - $QDSDD_0$ tensor.

Proof. If \mathcal{A} is a quasi-doubly strictly diagonally dominant tensor, then for $i, j \in N$, $j \neq i$, the strict inequality (11) holds, i.e.,

$$(|a_{i\dots i}| - r_i^j(\mathcal{A})) |a_{j\dots j}| > r_j(\mathcal{A}) |a_{ij\dots j}|.$$

For a given nonempty proper subset S of N , we easily get that for each $i \in S$ and each $j \in \bar{S}$,

$$(|a_{i\dots i}| - r_i^j(\mathcal{A})) |a_{j\dots j}| > r_j(\mathcal{A}) |a_{ij\dots j}|,$$

and

$$(|a_{j\dots j}| - r_j^i(\mathcal{A})) |a_{i\dots i}| > r_i(\mathcal{A}) |a_{ji\dots i}|.$$

Hence, \mathcal{A} is an S - $QDSDD$ tensor. Similarly, we can prove that a quasi-doubly diagonally dominant tensor is an S - $QDSDD_0$ tensor. \square

Proposition 3. Let $\mathcal{A} = (a_{i_1\dots i_m}) \in C^{[m,n]}$ and S be a nonempty proper subset of N . If \mathcal{A} is an S - $QDSDD$ tensor, then \mathcal{A} is an S - SDD tensor. If \mathcal{A} is an S - $QDSDD_0$ tensor, then \mathcal{A} is an S - SDD_0 tensor.

Proof. We only prove that an S - $QDSDD$ tensor is an S - SDD tensor, and by a similar way, we can prove that an S - $QDSDD_0$ tensor is an S - SDD_0 tensor.

Let \mathcal{A} be an S - $QDSDD$ tensor. It is easy to see from Definition 3 that either for any $i \in S$,

$$|a_{i\dots i}| > r_i(\mathcal{A}), \tag{15}$$

or for any $j \in \bar{S}$,

$$|a_{j\dots j}| > r_j(\mathcal{A}). \tag{16}$$

Without loss of generality, we next suppose that for any $j \in \bar{S}$, Inequality (16) holds. Hence, for any $j \in \bar{S}$,

$$|a_{j\dots j}| - r_j^i(\mathcal{A}) > |a_{ji\dots i}| \tag{17}$$

and

$$|a_{j\dots j}| - r_j^{\bar{\Delta}^S}(\mathcal{A}) > r_j^{\Delta^S}(\mathcal{A}). \tag{18}$$

Case I: for $i \in S$ such that Inequality (15) holds, i.e.,

$$|a_{i\dots i}| - r_i^{\bar{\Delta}^S}(\mathcal{A}) > r_i^{\Delta^S}(\mathcal{A}), \tag{19}$$

by combining with Inequalities (16) and (18) we easily get that for this $i \in S$ and each $j \in \bar{S}$, Inequalities (13) and (14) hold.

Case II: for $i \in S$ such that

$$|a_{i\dots i}| \leq r_i(\mathcal{A}),$$

by Definition 3 we can get that $0 < |a_{i\dots i}| \leq r_i(\mathcal{A})$,

$$0 < |a_{i\dots i}| - r_i^j(\mathcal{A}) \leq |a_{ij\dots j}|, \quad (20)$$

and

$$0 < |a_{i\dots i}| - r_i^{\bar{\Delta}^S}(\mathcal{A}) \leq r_i^{\Delta^S}(\mathcal{A}). \quad (21)$$

Let $a = |a_{i\dots i}|$, $b = r_i^{\bar{\Delta}^S}(\mathcal{A})$, $c = r_i^{\Delta^S}(\mathcal{A}) - |a_{ij\dots j}|$, $d = |a_{ij\dots j}|$, $e = |a_{j\dots j}|$, $f = r_j^{\bar{\Delta}^S}(\mathcal{A})$, $g = r_j^{\Delta^S}(\mathcal{A}) - |a_{ji\dots i}|$ and $h = |a_{ji\dots i}|$. If $r_j(\mathcal{A}) \neq 0$ for some $j \in \bar{S}$, then by Inequality (11), Inequality (20) and by Lemma 5, we have

$$\frac{|a_{i\dots i}| - r_i^{\bar{\Delta}^S}(\mathcal{A})}{r_i^{\Delta^S}(\mathcal{A})} \frac{|a_{j\dots j}|}{r_j(\mathcal{A})} \geq \frac{|a_{i\dots i}| - r_i^j(\mathcal{A})}{|a_{ij\dots j}|} \frac{|a_{j\dots j}|}{r_j(\mathcal{A})} > 1,$$

and $|a_{j\dots j}| \left(|a_{i\dots i}| - r_i^{\bar{\Delta}^S}(\mathcal{A}) \right) > r_j(\mathcal{A}) r_i^{\Delta^S}(\mathcal{A})$, i.e., Inequality (14) holds. Similarly, by Inequality (11) and by Lemma 5, we can also get

$$\frac{|a_{i\dots i}|}{r_i(\mathcal{A})} \frac{|a_{j\dots j}| - r_j^{\bar{\Delta}^S}(\mathcal{A})}{r_j^{\Delta^S}(\mathcal{A})} \geq \frac{|a_{i\dots i}| - r_i^j(\mathcal{A})}{|a_{ij\dots j}|} \frac{|a_{j\dots j}|}{r_j(\mathcal{A})} > 1,$$

where $\frac{|a_{j\dots j}| - r_j^{\bar{\Delta}^S}(\mathcal{A})}{r_j^{\Delta^S}(\mathcal{A})} = +\infty$ if $r_j^{\Delta^S}(\mathcal{A}) = 0$, and $|a_{i\dots i}| \left(|a_{j\dots j}| - r_j^{\bar{\Delta}^S}(\mathcal{A}) \right) > r_i(\mathcal{A}) r_j^{\Delta^S}(\mathcal{A})$, i.e., Inequality (13) holds. On the other hand, if $r_j(\mathcal{A}) = 0$ for some $j \in \bar{S}$, then $r_j^{\bar{\Delta}^S}(\mathcal{A}) = r_j^{\Delta^S}(\mathcal{A}) = 0$. Obviously, Inequalities (13) and (14) also hold. The conclusion follows from Cases I and II. \square

As shown in [13, 14], by using eigenvalue localization sets for tensors, one can give some corresponding checkable sufficient conditions of the positive (semi-)definiteness of tensors. Here we call a tensor $\mathcal{A} = (a_{i_1 \dots i_m}) \in R^{[m, n]}$ symmetric [20, 26] if

$$a_{i_1 \dots i_m} = a_{\pi(i_1 \dots i_m)}, \forall \pi \in \Pi_m,$$

where Π_m is the permutation group of m indices. And an even-order real symmetric tensor is called positive (semi-)definite, if its smallest H-eigenvalue is positive (nonnegative). Next, a new checkable sufficient condition of the positive (semi-)definiteness of tensors is obtained by using Theorem 4.

Theorem 7. *Let $\mathcal{A} = (a_{i_1 \dots i_m}) \in R^{[m, n]}$ with $n \geq 2$ and S be a nonempty proper subset of N . If \mathcal{A} is an even-order symmetric S -SDD (S -SDD₀) tensor with $a_{k \dots k} > (\geq) 0$ for all $k \in N$, then \mathcal{A} is positive (semi-)definite.*

Proof. We need only prove that \mathcal{A} is positive semi-definite, and by a similar way, we can prove that \mathcal{A} is positive definite. Let λ be an H-eigenvalue of \mathcal{A} . Suppose on the contrary that $\lambda < 0$. From Theorem 4, we have $\lambda \in \Omega^S(\mathcal{A})$ which implies that there are $i_0, i_1 \in S, j_0, j_1 \in \bar{S}$ such that $\lambda \in \Omega_{i_0, j_0}^S(\mathcal{A})$ or $\lambda \in \Omega_{j_1, i_1}^{\bar{S}}(\mathcal{A})$, that is,

$$|\lambda - a_{i_0 \dots i_0}| \left(|\lambda - a_{j_0 \dots j_0}| - r_{j_0}^{\bar{\Delta}^S}(\mathcal{A}) \right) \leq r_{i_0}(\mathcal{A}) r_{j_0}^{\Delta^S}(\mathcal{A})$$

or

$$|\lambda - a_{j_1 \dots j_1}| \left(|\lambda - a_{i_1 \dots i_1}| - r_{i_1}^{\bar{\Delta}^S}(\mathcal{A}) \right) \leq r_{j_1}(\mathcal{A}) r_{i_1}^{\Delta^{\bar{S}}}(\mathcal{A}).$$

On the other hand, since \mathcal{A} is an S -SDD₀ tensor with $a_{k \dots k} \geq 0$ for all $k \in N$, we have that for $i_0, i_1 \in S, j_0, j_1 \in \bar{S}$,

$$|\lambda - a_{i_0 \dots i_0}| \left(|\lambda - a_{j_0 \dots j_0}| - r_{j_0}^{\bar{\Delta}^S}(\mathcal{A}) \right) > |a_{i_0 \dots i_0}| \left(|a_{j_0 \dots j_0}| - r_{j_0}^{\bar{\Delta}^S}(\mathcal{A}) \right) \geq r_{i_0}(\mathcal{A}) r_{j_0}^{\Delta^S}(\mathcal{A})$$

and

$$|\lambda - a_{j_1 \dots j_1}| \left(|\lambda - a_{i_1 \dots i_1}| - r_{i_1}^{\bar{\Delta}^S}(\mathcal{A}) \right) > |a_{j_1 \dots j_1}| \left(|a_{i_1 \dots i_1}| - r_{i_1}^{\bar{\Delta}^S}(\mathcal{A}) \right) \geq r_{j_1}(\mathcal{A}) r_{i_1}^{\Delta^{\bar{S}}}(\mathcal{A}).$$

These lead to a contradiction. Hence, $\lambda \geq 0$, and \mathcal{A} is positive semi-definite. The conclusion follows. \square

According to Theorem 7, Proposition 1, Proposition 2 and Proposition 3, we easily obtain the following results which were also obtained in [5, 11, 13, 28].

Corollary 1. An even-order strictly diagonally dominant symmetric tensor with all positive diagonal entries is positive definite. And an even-order diagonally dominant symmetric tensor with all nonnegative diagonal entries is positive semi-definite.

Corollary 2. An even-order quasi-doubly strictly diagonally dominant symmetric tensor with all positive diagonal entries is positive definite. And an even-order quasi-doubly diagonally dominant tensor symmetric tensor with all nonnegative diagonal entries is positive semi-definite.

Corollary 3. An even-order S - $QDSDD$ tensor with all positive diagonal entries is positive definite. And an even-order S - $QDSDD_0$ symmetric tensor with all nonnegative diagonal entries is positive semi-definite.

Example 1. Let $\mathcal{A} = (a_{ijkl}) \in R^{[4,3]}$ be a real symmetric tensor with elements defined as follows:

$$\begin{aligned} a_{1111} &= 5, a_{2222} = 6, a_{3333} = 3.3, a_{1112} = -0.1, a_{1113} = 0.1, a_{1122} = -0.2, \\ a_{1123} &= -0.2, a_{1133} = 0, a_{1222} = -0.1, a_{1223} = 0.3, a_{1233} = 0.1, \\ a_{1333} &= -0.1, a_{2223} = 0.1, a_{2233} = -0.1, a_{2333} = 0.2. \end{aligned}$$

Let $S = \{1, 2\}$. Obviously $\bar{S} = \{3\}$. By computations, we get that

$$|a_{1111}| (|a_{3333}| - r_3^1(\mathcal{A})) = -0.5000 < 0.3800 = |a_{3111}| r_1(\mathcal{A}).$$

Hence, \mathcal{A} is not an S - $QDSDD_0$ tensor, consequently, not a strictly diagonally dominant symmetric tensor or a quasi-doubly strictly diagonally dominant tensor, and hence we can not use Corollary 1, Corollary 2 or Corollary 3 to determine the positiveness of \mathcal{A} . However, it is easy to get that

$$|a_{1111}| \left(|a_{3333}| - r_3^{\overline{\Delta^S}}(\mathcal{A}) \right) = 4.0000 > 3.8000 = r_1(\mathcal{A}) r_3^{\Delta^S}(\mathcal{A}),$$

$$|a_{3333}| \left(|a_{1111}| - r_1^{\overline{\Delta^S}}(\mathcal{A}) \right) = 4.2900 > 0.3500 = r_3(\mathcal{A}) r_1^{\Delta^S}(\mathcal{A}),$$

$$|a_{2222}| \left(|a_{3333}| - r_3^{\overline{\Delta^S}}(\mathcal{A}) \right) = 4.8000 > 4.5000 = r_2(\mathcal{A}) r_3^{\Delta^S}(\mathcal{A})$$

and

$$|a_{3333}| \left(|a_{2222}| - r_2^{\overline{\Delta^S}}(\mathcal{A}) \right) = 5.6100 > 0.7000 = r_3(\mathcal{A}) r_2^{\Delta^S}(\mathcal{A}),$$

i.e., \mathcal{A} is an S - SDD tensor. By Theorem 7, \mathcal{A} is positive definite.

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